# Plane waves, integrable quantum systems and the group $U(p, q)$, $p \leqslant q$



1997 J. Phys. A: Math. Gen. 30 4109

(http://iopscience.iop.org/0305-4470/30/11/034)

J. Phys. A: Math. Gen. **30** (1997) 4109–4116. Printed in the UK PII: S0305-4470(97)79312-0

# Plane waves, integrable quantum systems and the group $U(p, q)$, $p \leqslant q$

Yi A Verdiyev†

Trakya University Faculty of Arts and Sciences Department of Mathematics Edirne, Turkey
International Center for Physics and Applied Mathematics, Edirne, Turkey

Received 5 November 1996

**Abstract.** Plane waves on symmetric spaces (SS) $X_c \equiv SU(p,q)/S(U(p) \otimes U(q))$, $p \leqslant q = 2, 3, \ldots,$ are constructed. The integrable $n$-body quantum systems related to SS $X$ are considered.

## 1. Introduction

This paper is a continuation of the paper concerning the plane waves and integrable quantum systems of the group $SO(p,q)$ [1]. Hence, we refer to this reference for details. Here we have considered the spaces related with Cartan involutive automorphism of the group $U(p,q)$, $p \leqslant q = 2, 3, \ldots,$ namely the symmetric Riemannian and pseudo-Riemannian spaces with rank equal to $p = 1, 2, \ldots$

$$X_c \equiv SU(p,q)/S(U(p) \otimes U(q)) \qquad Z_c \equiv U(p,q)/U(1) \otimes U(p-1,q).$$

We construct the plane waves on SS $X_c$ and $Z_c$ and calculate Harish–Chandra's $c$-functions. Here we give an example of the integrable quantum system related to SS, $SU(2,2)/S(U(2) \otimes U(2))$.

The paper is organized as follows. In section 2 for completeness and to fix the notation we construct the plane waves on SS with rank 1 and give the expression for the Harish–Chandra $c$-functions. In section 3 the main result on construction of the plane waves on SS with rank $p > 1$ and consideration of integrable quantum systems related to SS $SU(2,2)/S(U(2) \otimes U(2))$ are presented.

## 2. Plane waves on SS of rank 1 of the group $U(p, q)$, $p \leqslant q$

In this section we construct plane waves on SS defined by quadratic forms in the space $C^{p,q}$.

### *2.1. Plane waves on SS of the group $U(1,q)$*

We use the notation $z = (z_0, z_1, \ldots, z_q)$ for the elements of an $(n = 1 + q)$-dimensional complex vector space $C^{1,q}$. In $C^{1,q}$ we define the bilinear product $[z, z'] = z_0\bar{z}'_0 - z_1\bar{z}'_1 -$

† On leave of absence from the Physical Institute of the Academy of Science of Azerbaijan Republic.

$\cdots - z_q\bar{z}'_q$. Points $z$ such that $[z, z] = 1$ form a complex hyperboloid $H_c^q$ in $C^{1,q}$. The stabilizer of $\overset{\circ}{z} = (1, 0, \ldots, 0) \in H_c^q$ coincides with the subgroup $U(q)$. Therefore,

$$H_c^q \approx U(1, q)/U(q). \tag{2.1}$$

Every point $z \in H_c^q$ is representable in the form

$$z = (\mathrm{e}^{\mathrm{i}\varphi_0}\varsigma_0, \mathrm{e}^{\mathrm{i}\varphi_1}\varsigma_1, \ldots, \mathrm{e}^{\mathrm{i}\varphi_q}\varsigma_q) \qquad \text{where } 0 \leqslant \varphi_i < 2\pi, \ \varsigma_i \geqslant 0$$
$$i = 0, 1, 2, \ldots, q, \ \varsigma = (\varsigma_1, \ldots, \varsigma_q) \tag{2.2}$$

is the point of the real hyperboloid $[\varsigma, \varsigma] = \varsigma_0^2 - \cdots - \varsigma_q^2 = 1$. From $\cosh kr = \mathrm{Re}[z, \overset{\circ}{z}] = \cos\varphi_0\varsigma_0$ it follows that the distance has real and imaginary parts. Hence the space $H_c^q$ is a symmetric pseudo-Riemannian space. This fact can be seen by identifying the space $C^{1,q}$ with the pseudo-Euclidean space $R^{2,2q}$. Then we have

$$U(1, q)/U(q) \approx SO(2, 2q)/SO(1, 2q). \tag{2.3}$$

By identifying the points $\mathrm{e}^{\mathrm{i}\varphi}z$, $0 \leqslant \varphi < 2\pi$, of $H_c^q$ we have obtained a symmetric Riemannian space:

$$X_c \approx U(1, q)/U(1) \times U(q) \approx SU(1, q)/S(U(1) \times U(q)). \tag{2.4}$$

Indeed, the transitive motion group of $X$ coincides with $SU(1, q)$. The subgroup $S(U(1) \times U(q))$ is the stabilizer of the point $\overset{\circ}{x} = (\mathrm{e}^{\mathrm{i}\varphi}, 0, \ldots, 0) \in X_c$. Every point $x \in X_c$ is representable of the form

$$x = (\varsigma_0, \mathrm{e}^{\mathrm{i}\varphi_1}\varsigma_1, \ldots, \mathrm{e}^{\mathrm{i}\varphi_q}\varsigma_q). \tag{2.5}$$

The distance between points $\overset{\circ}{x} = (1, 0, \ldots, 0)$ and $x$ defined by the formula

$$\cosh kr = \mathrm{Re}[\overset{\circ}{x}, x] = \varsigma_0 \geqslant 1. \tag{2.6}$$

Points $y \in C^{1,q}$ for which $[y, y] = 0$, form the complex cone $Y_c$ in $C^{1,q}$. Every point $y \in C^{1,q}$ is representable in the form

$$y = (\mathrm{e}^{\mathrm{i}\varphi_0}\nu_0, \mathrm{e}^{\mathrm{i}\varphi_1}\nu_1, \ldots, \mathrm{e}^{\mathrm{i}\varphi_q}\nu_q) \qquad \text{where } 0 \leqslant \varphi_i \leqslant 2\pi, \ \nu_i \geqslant 0, \ i = 0, 1, \ldots, q. \tag{2.7}$$

$\nu = (\nu_0, \nu_1, \ldots, \nu_q)$ is the point real cone $Y$:$[\nu, \nu] = 0$.

The maximal degenerate representation of the group $U(1, q)$ can be constructed in the space $D_\sigma$ of infinitely differentiable homogeneity functions $F(y)$ with homogeneously degree $\sigma$ on a cone $Y_c$ [2]:

$$F(ay) = a^\sigma F(y) \qquad a > 0 \qquad y \in Y_c \tag{2.8}$$

and with condition $F(uy) = F(y)$, $|u| = 1$. The representation

$$T_\sigma(g)F(y) = F(yg) \qquad g \in SU(1, q) \tag{2.9}$$

can be realized in the space of infinitely differentiable functions on intersections of the cone, for example on the complex sphere $S_c^{q-1} = SU(q)/SU(q-1)$:

$$T_\sigma(g)f(s) = |(sg)|^\sigma f(\overline{sg}) \qquad g \in SU(1, q) \tag{2.10}$$

where $s = y|_{y_0=1} = (1, \eta)$, $y = \mathrm{e}^\alpha \mathrm{e}^{\mathrm{i}\varphi_0}s$, $\eta \in S_c^{q-1}$, $\overline{sg} = sg/(sg)_0$.

From $y' = yg$ we have

$$|(sg)_0| = \mathrm{e}^{\alpha'-\alpha}\overline{sg} = (1, \eta_g). \tag{2.11}$$

The representation $T_\sigma(g)$, $g \in SU(1, q)$, is a unitary representation with respect to the scalar product

$$\langle f_1, f_2\rangle = \int f_1(\eta)\overline{f_2(\eta)}\,\mathrm{d}\eta \tag{2.12}$$

where $\mathrm{d}\eta$ is an invariant volume on $S_c$, if $\sigma = -q + \mathrm{i}\rho$, $\rho \in [0, \infty)$ (the principal series). Indeed from the relations

$$y' = yg \Rightarrow y_0' = y_0(sg)_0 \Rightarrow \mathrm{e}^{\alpha'}\mathrm{e}^{\mathrm{i}\varphi_0'} = \mathrm{e}^{\alpha}\mathrm{e}^{\mathrm{i}\varphi_0}(sg)_0$$

and from the invariant of the volume element on $Y_c$, $\mathrm{d}y = \mathrm{d}y'$, where $\mathrm{d}y = \mathrm{e}^{2q\alpha}\mathrm{d}\alpha\,\mathrm{d}\varphi_0\,\mathrm{d}\eta$, we have

$$\mathrm{d}\eta = |(sg)_0|^{2q}\mathrm{d}\eta'. \tag{2.13}$$

It follows that the invariant condition of the scalar product $\langle f_1, f_2\rangle$ under the representation equation (2.10) is $\sigma + \overline{\sigma} + 2q = 0$. The representations $T_\sigma(g)$ and $D_{-2q-\sigma}(g)$, $g \in SU(1, q)$ are equivalent.

As in the case of the group $SO(1, q)$ [1] we find that the zonal spherical functions of the representation $T_\sigma(g)$, $g \in SU(1, q)$ are given by the formula

$$T^{\sigma}_{O_k;O_k}(g_x) = \int_{S_0^{q-1}} |[x, s]|^{\sigma}\,\mathrm{d}\eta \qquad x \in X_c. \tag{2.14}$$

Here $|O_k\rangle$ is the invariant vector with respect to the representation

$$T_\sigma(k),\ \ k \in U(1) \times U(q).$$

The plane waves $|[x, s]|^{\sigma}$ are the eigenfunctions of the Laplace–Beltrami operator on the SS $X_c$:

$$\Delta_{L,B}|[x, s]|^{\sigma} = -\sigma(\sigma + 2q)|[x, s]|^{\sigma}. \tag{2.15}$$

The zonal spherical functions $T^{\sigma}_{O_k;O_k}(a(\alpha))$ satisfy the equation

$$\left(\frac{1}{\sinh^{2q}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\sinh^{2q}\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)t^{\sigma}_{O_k;O_k}(a(\alpha)) = \sigma(\sigma + 2q)t^{\sigma}_{O_k;O_k}(a(\alpha)) \tag{2.16}$$

for the Harish–Chandra $c$-function from (2.14) we have the integral representation

$$C(\sigma) = \int_0^{2\pi}\int_0^{2\pi} |1 - \mathrm{e}^{\mathrm{i}\varphi}\cos\theta|^{\sigma}\sin^{2q-3}\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{2.17}$$

Calculation of this integral was done by Helgason [3]:

$$C(\sigma) = \frac{\Gamma(-\sigma - q)}{[\Gamma(-\sigma/2)]^2}. \tag{2.18}$$

The orthogonality and completeness conditions for the plane waves on SS $X_c$ of the group $U(1, 2)$ are similar to those of the plane waves on SS $X$ of the group $SO(1, q)$.

To construct the plane waves on SS $H_c^q$ we consider the representation $T_{\sigma,m}(g)$, $g \in U(1, q)$ in the space $D_{\sigma,m}$ of the functions $F(y)$, $y \in Y_c$ with the condition [2]

$$F(uy) = u^m F(y) \qquad |u| = 1 \tag{2.19}$$

where $m$ is an integer. The zonal spherical function of the representation $T_{\sigma,m}(g)$, $g \in U(1, q)$, has the form

$$t^{\sigma,m}_{O_k;O_k}(d_0(\mathrm{e}^{\mathrm{i}\varphi})a(\alpha)) = \mathrm{e}^{-\mathrm{i}m\varphi}t^{\sigma,m}_{O_k;O_k}(a(\alpha)) \tag{2.20}$$

where $d_0(\mathrm{e}^{\mathrm{i}\varphi}) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi}, 1, \ldots, 1) \in U(1)$, $|O_k\rangle$ is an invariant vector with respect to the representation $T_{\sigma,m}(k)$, $k \in U(q)$. From equation (2.19) it follows that

$$t^{\sigma,k}_{O_k;O_k}(g_z) = \int_{S_c^{q-1}} [z, s]^{(\sigma+m)/2}\overline{[z, s]}^{(\sigma-m)/2}\,\mathrm{d}\eta. \tag{2.21}$$

The plane waves $[z,s]^{(\sigma+m)/2}\overline{[z,s]}^{(\sigma-m)/2}$ are eigenfunctions of the Laplace–Beltrami operator on SS $H_c^q$. The zonal spherical functions $t^{\sigma}_{O_k;O_k}(a(\alpha))$ satisfy the equation

$$\left(\frac{1}{\sinh^{2q-1}\alpha\cosh\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\sinh^{2q-1}\alpha\cosh\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}+\frac{m^2}{\cosh^2\alpha}\right)t^{\sigma,m}_{O_k;O_k}(\alpha)=\sigma(\sigma+1)t^{\sigma,m}_{O_k;O_k}(\alpha). \tag{2.22}$$

For the Harish–Chrandra $c$-functions from equation (2.21) we obtain the integral representation

$$C_m(\alpha)=\int_0^{2\pi}\int_0^{2\pi}(1-\mathrm{e}^{\mathrm{i}\varphi}\cos\theta)^{(\sigma+m)/2}(1-\mathrm{e}^{\mathrm{i}\varphi}\cos\theta)^{(\sigma-m)/2}\sin^{2q-3}\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{2.23}$$

The expression of the $C_m(\sigma)$-function has the form [2]

$$C_m(\sigma)=\frac{\Gamma(-\sigma-q)}{\Gamma((-\sigma-m)/2)\Gamma((-\sigma+m)/2)}. \tag{2.24}$$

The substitution

$$T^{\sigma,m}_{O_k;O_k}(\alpha)=\sinh^{-q+1/2}\alpha\cosh^{-1/2}\alpha\Psi(\alpha) \tag{2.25}$$

reduces (2.22) to the Schrödinger equation with potential

$$V=[(q-1)^2-\tfrac{1}{4}]\sinh^{-2}\alpha-(m^2-\tfrac{1}{4})\cosh^{-2}\alpha. \tag{2.26}$$

The orthogonality conditions for plane waves on SS $H_c^q$ are similar to formula (2.18) from [1] for the plane waves on SS $X$ of the group $SO(1,q)$, but in the completeness condition over $\sigma$ there is a summation over the integer $m$.

### 2.2. Plane waves on SS of the group $U(p,q)$

In the complex vector space $C^{p,q}$ we define the bilinear product

$$[z,z']=z_1\overline{z'_1}+\cdots+z_p\overline{z'_p}-z_{p+1}\overline{z'_{p+1}}-\cdots-z_{p+q}\overline{z'_{p+q}}.$$

The point $z$ such that $[z,z]=\pm1$ forms complex hyperboloids $H_c^{p-1,q}$ and $H_c^{p,q-1}$ in $C^{p,q}$, respectively. Every point $H_c^{p-1,q}$ (or $H_c^{p,q-1}$) is representable in the form

$$z=(\mathrm{e}^{\mathrm{i}\varphi_1}\varsigma_1,\ldots,\mathrm{e}^{\mathrm{i}\varphi_p}\varsigma_p,\mathrm{e}^{\mathrm{i}\varphi_{p+1}}\varsigma_{p+1},\ldots,\mathrm{e}^{\mathrm{i}\varphi_{p+q}}\varsigma_{p+q}) \tag{2.27}$$

where $0\leqslant\varphi_i\leqslant2\pi$, $\varsigma_i\geqslant0$, $i=1,\ldots,p+q$, $\varsigma=(\varsigma_1,\ldots,\varsigma_p,\varsigma_{p+1},\ldots,\varsigma_{p+q})$ is the point of the real hyperboloid $z_+$ (or $z_-$): $[\varsigma,\varsigma]=+1$ (or $[\varsigma,\varsigma]=-1$). The points $y\in C^{p,q}$ for which $[y,y]=0$ form a complex cone $Y_c$ in $C^{p,q}$. Every point $y\in Y_c$ is representable in the form

$$y=(\mathrm{e}^{\mathrm{i}\varphi_1}\nu_1,\ldots,\mathrm{e}^{\mathrm{i}\varphi_p}\nu_p,\mathrm{e}^{\mathrm{i}\varphi_{p+1}}\nu_{p+1},\ldots,\mathrm{e}^{\mathrm{i}\varphi_{p+q}}\nu_{p+q}) \tag{2.28}$$

where $0\leqslant\varphi_1\leqslant2\pi$, $\nu_1\geqslant0$, $i=1,\ldots,p+q$, $\nu=(\nu_1,\ldots,\nu_p,\nu_{p+1},\ldots,\nu_{p+q})$ is the point real cone $Y$:$[\nu,\nu]=0$.

As in the case of the group $SO(p,q)$ [1] we find that the zonal spherical functions of the representation $T_\sigma(g)$, $g\in U(p,q)$ in the mixed basis are given by the formula

$$T_{O_H;O_K}(g_z)=\int_{S_c^{p-1}\times S_c^{q-1}}|[z,s]|^{\sigma}\,\mathrm{d}\eta. \tag{2.29}$$

Here $|O_k\rangle$ is the invariant vector with respect to the representation $t_\sigma(k)$, $k\in U(p)\times U(q)$ and $|O_H\rangle$ is the invariant vector with respect to the representation $T_\sigma(h)$, $h\in H\approx$

$U(p-1,q)$ (or $U(p,q-1)$). The points $z \in H_c^{p-1,q}$ (or $H_c^{p,q-1}$) and $s \in Y_c$ are represented in the bispherical coordinate systems on $H_c^{p-1,q}$ (or $H_c^{p,q-1}$) and $Y_c$:

$$z = (\cosh\alpha\eta^{(p)}, \sinh\alpha\eta^{(q)}) \qquad \text{or} \qquad z = (\sinh\alpha\eta^{(p)}, \cosh\alpha\eta^{(q)}) \tag{2.30}$$

and $s = (\eta^{(p)}\eta^{(q)})$, $\eta^{(p)} \in S_c^{p-1}$, $\eta^{(q)} \in S_c^{q-1}$, respectively. The zonal spherical functions $T^\sigma_{O_H;O_K}(g_z) = T^\sigma_{O_H;O_K}(g(\alpha))$, $z \in H_c^{p-1,q}$ satisfy the equation

$$\left(\frac{1}{\sinh^{2p-1}\alpha\cosh^{2q-1}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\sinh^{2p-1}\alpha\cosh^{2q-1}\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)T^\sigma_{O_H;O_K}(a(\alpha)) \\ = \sigma(\sigma+2p+2q-2)T^\sigma_{O_H;O_K}(a(\alpha)). \tag{2.31}$$

As in the case of the group $SO(p,q)$ for the Harish–Chandra $c$-function we have

$$C^{p,q}(\sigma) = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} |\cos\theta - \mathrm{e}^{\mathrm{i}\varphi}\cos\omega|^{-\sigma-2p+2q+2}\sin^{2p-3}\theta\cos\theta\sin^{2q-3} \\ \times\omega\cos\omega\,\mathrm{d}\theta\,\mathrm{d}\omega\,\mathrm{d}\varphi. \tag{2.32}$$

By reparametrization, this integral can be simplified [4] and we have

$$C^{p,q}(\sigma) = \int_0^{\pi}\int_0^{\pi} |\cos\theta - \cos\omega|^{-\sigma-2p+2q+2}\sin^{2p-2}\theta\sin^{2q-2}\omega\,\mathrm{d}\theta\,\mathrm{d}\omega \\ = \frac{\Gamma(-\sigma-p-q+1)\Gamma((-\sigma+3/2)-p-q)}{\sqrt{\pi}\,\Gamma(-\sigma/2)\Gamma((-\sigma/2)-p+1)\Gamma((-\sigma/2)-q+1)}. \tag{2.33}$$

Note that this expression for the $c$-function can be obtained from the expression of the $c$-function of the group $SO(p,q)$ (see equation (2.63) from [1]) by the substitution $p \to 2p$, $q \to 2q$.

## 3. Plane waves on SS of rank $p > 1$ of the group $U(p,q)$

From the definition $gI\bar{g}^t = I$, $g \in U(p,q)$ of the pseudo-unitary group $U(p,q)$ it follows that the elements of the maximal compact subgroup $K = U(p) \times U(q)$ is fixed under the involutive automorphism $\sigma(g) = IgI$. The symmetric Riemannian space $X_c = U(p,q)/U(p)\times U(q)$ with fixed point $\mathring{x} = 1$-unit has dimension $d = (p+q)^2 - p^2 - q^2 = 2pq$. The centralizer subgroup $M$ of $A$ in $K$ has the form

$$M = \mathrm{diag}(u_1,\ldots,u_p,U(q-p),u_p,\ldots,u_1), u_i\varepsilon U(1). \tag{3.1}$$

For the Cartan decomposition $G = K'AK$ of the group $U(p,q)$ we have

$$x = ka^2k^{-1} \qquad k \in K,\ a \in A. \tag{3.2}$$

The $a(\alpha^1,\ldots,\alpha^p) = \prod_{j=1}^p a(\alpha^j)$ are hyperbolic rotations in planes $(x^j, x^{p+q+1-j})$, $j = 1,\ldots,p$. We define the cone $Y_c$ with fixed point $\mathring{y}_1$ (see equation (3.9) from [1]) by the formula

$$y_1 = g\mathring{y}_1\sigma(g^{-1}). \tag{3.3}$$

The stabilizer of the fixed point $\mathring{y}_1$ in $U(p,q)$ coincides with the semidirect product of the subgroup $U(p-1,q-1)$ and $N$ consists of the matrices

$$n = \begin{pmatrix} 1-\mathrm{i}\gamma+z^2/2, z_1,\ldots,z_{p-1},z_p,\ldots,z_{p+q-2}, \mathrm{i}\gamma - z^2/2 \\ -z_1 & & z_1 \\ -z_{p-1} & I & z_{p-1} \\ z_p & & -z_p \\ z_{p+q-2} & & -z_{p+q-2} \\ -\mathrm{i}\gamma+z^2/2, z_1,\ldots,z_{p-1},z_p,\ldots,z_{p+q-2}, 1+\mathrm{i}\gamma-z^2/2 \end{pmatrix} \tag{3.4}$$

where $I$ is the $(p+q-2)\times(p+q-2)$ unit matrix, $z^2 = z_1^2+\cdots+z_{p-1}^2 - z_p^2 - \cdots - z_{p+q-2}^2$ and $\gamma$ is a real parameter.

For the Iwasava decomposition we have

$$y_1 = \mathrm{e}^{2\alpha^1} k \overset{\circ}{y}_1 k^{-1} \qquad y, \overset{\circ}{y} \in Y_c \qquad k \in U(p)U(q). \tag{3.5}$$

The metric on the SS $X_c$ is defined by the formula

$$[x_1, x_2] = \tfrac{1}{2}\,\mathrm{tr}(I x_1 I \bar{x}_2^t) \qquad x_1, x_2 \in X_c. \tag{3.6}$$

For the metric matrix in the SS $X_c$ we have

$$g_{ij} = \frac{1}{n}\,\mathrm{tr}(I\dot{x}_{\tau_i}, \overline{\dot{x}_{\tau_j}^t}) \tag{3.7}$$

where $\dot{x}_{\tau_i} = \mathrm{d}x/\mathrm{d}\tau_i$, $\tau_i$ are coordinates of the SS. The radial part of the Laplace–Beltrami operator on SS is defined by the formula [5]

$$\frac{1}{\sqrt{g}}\sum_{j=1}^{p}\frac{\partial}{\partial x^j}\sqrt{q}\frac{\partial}{\partial x^j}$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$. For $\sqrt{g}$ we have

$$\sqrt{g} = \prod_{i=j}^{p} \sinh^2(\alpha^i - \alpha^j)\sinh^2(\alpha^i + \alpha^j)\sinh^{2(q-p)}\alpha^i \sinh 2\alpha^i. \tag{3.8}$$

Here Cartan coordinates $\alpha^i - \alpha^j$, $\alpha^i + \alpha^j$, $\alpha^i$, $2\alpha^i$ of the group $U(p,q)$ correspond to the positive restricted roots $\omega^i - \omega^j$, $\omega^i + \omega^j$, $\omega^i$, $2\omega^i$ of the algebra of the group $U(p,q)$ with multiplicity 2, 2, $q-p$ and 1, respectively. The irreducible representations $T_{\chi_1}(g)$, $\chi_1 = (\sigma_1, \chi_2)$, $g \in U(p,q)$, of the group $U(p,q)$ we construct in the space $D_{\chi_1}$ of an infinitely differentiable vector function with homogeneity degree $\sigma_1$ and with the condition $F(uy) = F(y)$, $u \in U(1)$, whose values belong to the space $D_{\chi_2}$ of the representation $T_{\chi_2}(\tilde{g})$, $\tilde{g} \in SU(p-1, q-1)$. The representation formula can be written in the form

$$T_\chi(g) f(s_1) = \mathrm{e}^{(\alpha'_g - \alpha')\sigma_1} t^{\chi_2}(\tilde{g}(s_1, g)) f(\overline{g s_1 \sigma(g^{-1})}) \qquad g \in U(p,q) \tag{3.9}$$

where $y_1 = \mathrm{e}^{\alpha'} s_1$, $s_1 = k\overset{\circ}{y}_1 k^{-1}$, $\chi_1 = (\sigma_1, \chi_2)$, $\tilde{g}$ is the element of the stability subgroup $U(p-1, q-1)$; $f(s)$ is a vector function on the intersection of the cone $s_1 = k\overset{\circ}{y}_1 k^{-1}$ with values in the representation space $D_{\chi_2}$ of the stability subgroup $U(p-1, q-1)$. The expressions $\mathrm{e}^{(\alpha'_g - \alpha')\sigma_1}$ and $\tilde{g}(s_1, g)$ are defined from the relations

$$y_1' = g y_1 \sigma(g^{-1}) \qquad \mathrm{e}^{(\alpha^1_g - \alpha^1)} k'^{-1} g k = n\tilde{g}(s_1, g). \tag{3.10}$$

It follows that

$$\mathrm{e}^{2(\alpha'_g - \alpha')} = \tfrac{1}{2}|\,\mathrm{tr}(g s_1 \sigma(g^{-1})|. \tag{3.11}$$

The unitary representation $T_{\chi_1}(g)$, $g \in U(p,q)$, with respect to the scalar product $(f_1, f_2) = \int\langle\overline{f_1(s)} f_2(s)\rangle\,\mathrm{d}s$ is defined by $\sigma_1 = -(p+q) + 1 + \mathrm{i}\rho_1$, $\rho_1 \in [0, \infty]$ (principal series). Here $\langle\,\rangle$ denotes the inner product of vector-valued functions in the space $D_{\chi_2}$

$$\mathrm{d}s = \mathrm{d}\varsigma^{(p)}\mathrm{d}\xi^{(q)} \qquad \xi^{(q)} \in U(p)/U(p-1) \qquad \varsigma^{(p)} \in U(q)/U(q-1).$$

For the element $g_x$, which is defined by $x = g_x \overset{\circ}{x} \sigma(g_x^{-1})$, from (3.11) it follows that

$$\mathrm{e}^{2(\alpha'_{g_x^{-1}} - \alpha')} = \tfrac{1}{2}|\,\mathrm{tr}(I k_x a_x^2 k_x^{-1} I s_1)| = |[x, s_1]|. \tag{3.12}$$

For simplicity let us consider the case of the group $U(2,2)$. Concerning the physics of the eight-dimensional SS $X = U(2,2)/U(2)\times U(2)$, see the report [6] which was given by

A O Barut in the conference on 'Noncompact Lie Groups and some of their Applications'. In this case the zonal spherical functions on SS $X_c = U(2,2)/U(2)\times U(2)$ are defined by the formula

$$t^{\chi}_{O_k;O_k}(Ig_\chi) = \langle T_\chi(Ig_x^{-1})1, 1\rangle = \int |[x, s_1]|^{\sigma_1/2} t^{\sigma_2}_{O_k;O_k}(\bar{g}_x)\,\mathrm{d}s_1 \qquad (3.13)$$

where $\sigma_1 = -3+\mathrm{i}\rho_1$, $\sigma_2 = -1+\mathrm{i}\rho_2$, $\rho_1, \rho_2 \in [0,\infty)$. For the zonal spherical function $t^{\sigma_2}_{O_k;O_k}$ of the stability subgroup $U(1,1)$ we have a similar integral representation (see section 3 from [1]). So we have

$$t^{\chi}_{O_k;O_k}(g_\chi) = \int_{s_c^2\times s_c^2}\int_{s_c^1\times s_c^1} |[x, s_1]|^{(\sigma_1-\sigma_2)/2} |[x_1 k s_2 k^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)|^{\sigma_2}\,\mathrm{d}s_1\,\mathrm{d}s_2. \qquad (3.14)$$

The plane waves

$$|[x, s_1]|^{(\sigma_1-\sigma_2)/2} |[x_1 k s_2 k^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)|^{\sigma_2} \qquad (3.15)$$

are eigenfunctions of the Laplace–Beltrami operator on SS $X_c = U(2,2)/U(2)\times U(2)$. Harish–Chandra's $c(\sigma_1,\sigma_2)$-functions are represented in the form $c(\sigma_1,\sigma_2) = c_{2,2}c_{1,1}$, where the expressions of $c_{2,2}$ and $c_{1,1}$ are defined by formula (2.33).

Consider the quantum integrable system related to SS $X = U(2,2)/U(2)\times U(2)$.

$$x = k(\varphi_1, \theta_1, \psi_1\varphi_2, \theta_2, \psi_2)a^2(\alpha_1,\alpha_2)k^{-1}(\varphi_1, \theta_1, \psi_1\varphi_2, \theta_2\psi_2) \qquad (3.16)$$

where

$$k = \begin{pmatrix} (a_1/-b_1)(b_1/a_1) & 0 \\ 0 & (a_2/-b_2)(b_2/a_2) \end{pmatrix} \qquad a_j = \cos\frac{\theta_j}{2}\mathrm{e}^{\mathrm{i}(\varphi_j+\psi_j)}$$

$$b_j = \sin\frac{\theta_j}{2}\mathrm{e}^{\mathrm{i}(\varphi_j-\psi_j)} \qquad j = 1, 2$$

$$a(\alpha_1,\alpha_2) = \begin{pmatrix} \cosh\alpha_1 & 0 & 0 & \sinh\alpha_1 \\ 0 & \cosh\alpha_2 & \sinh\alpha_2 & 0 \\ 0 & \sinh\alpha_2 & \cosh\alpha_2 & 0 \\ \sinh\alpha_1 & 0 & 0 & \cosh\alpha_1 \end{pmatrix}.$$

Calculating the metric matrix by the formula

$$g_{ij} = \tfrac{1}{2}\,\mathrm{tr}(Ix_{t_i} Ix_{t_j}) \qquad i, j = 1, \ldots, 8$$

where $t_i$ are parameters of the SS $X$, we have

$$\sqrt{g} = \sinh^2(\alpha_1+\alpha_2)\sinh^2(\alpha_1-\alpha_2)|\sinh 2\alpha_1|\,|\sinh 2\alpha_2|. \qquad (3.17)$$

The radial part of the Laplace equation has the form

$$\left[\sum_{j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\alpha_j}\sqrt{g}\frac{\partial}{\partial\alpha_j} - \frac{l_j(l_j+1)}{\sinh^2 2\alpha_j}\right]\vartheta(\alpha_1,\alpha_2) = [\sigma_1(\sigma_1+6)+\sigma_2(\sigma_2+2)]\vartheta(\alpha_1,\alpha_2) \quad (3.18)$$

where $l(l+1)$ is the eigenvalue of the Laplace operator on the group $SU(2)$. By the substitution

$$\vartheta(\alpha_1,\alpha_2) = \sinh^{-1}(\alpha_1+\alpha_2)\sinh^{-1}(\alpha_1-\alpha_2)\omega(\alpha_1,\alpha_2) \qquad (3.19)$$

we get

$$\left[\frac{1}{4}\sum_{j=1}^{2}\frac{\partial^2}{\partial\alpha_j^2} + \frac{1}{2}\coth 2\alpha_j\frac{\partial}{\partial\alpha_j} - \frac{\frac{1}{4}l_j(l_j+1)}{\sinh^2 2\alpha_j}\right]\omega(\alpha_1,\alpha_2)$$
$$= \frac{1}{4}[\sigma_1(\sigma_1+6)+\sigma_2(\sigma_2+2)+8]\omega(\alpha_1,\alpha_2). \qquad (3.20)$$

The separation in the variables $\alpha_1$ and $\alpha_2$ of the solutions $\omega(\alpha_1, \alpha_2)$ of the equation is evident and we give rise to the eigenvalue problems of the radial part of the Laplace operator on a two-dimensional two-sheeted hyperboloid [7] on each variable $\tau_1 = 2\alpha_1$ and $\tau_2 = 2\alpha_2$. Finally, the substitution

$$\omega(\alpha_1, \alpha_2) = \sinh^{-1/2} 2\alpha_1 \sinh^{-1/2} 2\alpha_2 \psi(\alpha_1, \alpha_2) \tag{3.21}$$

reduces equations (3.20) to the two-particle one-dimensional Schrödinger equation with potentials

$$V(\tau_1, \tau_2) = \sum_{j=1}^{2} \frac{\frac{1}{4} l_j(l_j+1) - \frac{1}{4}}{\sinh^{-2} \tau_j}$$
$$E = -\tfrac{1}{4}\sigma_1(\sigma_1+6) - \tfrac{1}{4}\sigma_2(\sigma_2+2) - \tfrac{10}{4} = (\tfrac{1}{2}\mathrm{i}\rho)^2 + (\tfrac{1}{2}\mathrm{i}\rho_2)^2 \tag{3.22}$$

where $\tau_j = 2\alpha_j$, $\sigma_1 = -3 + \mathrm{i}\rho_1$, $\sigma_2 = -1 + \mathrm{i}\rho_2$. The $\psi$-function is represented in the form

$$\Psi(\tau_1, \tau_2) = \frac{4}{\rho_1^2 - \rho_2^2} [\Psi_{\rho_1}(\tau_1)\Psi_{\rho_2}(\tau_2) - \Psi_{\rho_1}(\tau_2)\Psi_{\rho_2}(\tau_1)] \tag{3.23}$$

with

$$\Psi(2\alpha_j, \rho_j/2) = \frac{\Gamma(\frac{1}{2} + n_j + \mathrm{i}\rho_j/2)}{\Gamma(\frac{1}{2} + \mathrm{i}\rho_j/2)\Gamma(n_j+1)} (\sinh \alpha_j)^{\frac{1}{2}+n} (\cosh \alpha_j)^{\frac{1}{2}-n}$$
$$\times F(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}\rho_j, \tfrac{1}{2} - \tfrac{1}{2}\mathrm{i}\rho_j, n+1; -\sinh^2 \alpha_j) \tag{3.24}$$

where $\sqrt{E_j} = \rho_j$ and $n_j = \frac{1}{2}\sqrt{l_j(l_j+1)}$, $j = 1, 2$.

These solutions and the general one's related to SS $X = U(p,q)/U(p) \times U(q)$ were found by Berezin and Karplevich [8].

## Acknowledgments

I am grateful to my assistant M Sezgin for helping me in the preparation of this paper and I H Duru and G A Kerimov for discussions. The support of the research from TÜBITAK is gratefully acknowledged.

## References

[1] Verdiyev Yi A 1997 Plane waves, integrable quantum systems, Green functions and the groups $SO(p,q)$, $p \leqslant q$ *J. Phys. A: Math. Gen.* **30** 4089

[2] Vilenkin N Ya and Klimyk A U 1993 *Representation of Lie Groups and Special Functions* (Dortrecht: Kluwer)

[3] Helgason S 1974 Eigenspaces of the Laplacian, integral representations and irreducibility *J. Funct. Anal.* **17** 328–53

[4] Facaut J 1979 Distribution spherique sur les espaces hyperboliques *J. Math. Pure Appl.* **58** 369–444

[5] Helgason S 1984 *Groups and Geometric Analysis* (New York: Academic)

[6] Barut A O 1994 *Noncompact Lie Groups and some of their Applications (NATO ASI, Series C, 429)* ed A Tanner and R Wilson (Dortrecht: Kluwer) p 103

[7] Dane C and Verdiyev Y A 1996 Integrable systems of group $SO(1,2)$ and Green functions *J. Math. Phys.* **37** 39

[8] Berezin F A and Karpelevich F I 1958 Zonal spherical functions and Laplace operators on some symmetries spaces *Dokl. Akad. Nauk. SSSR* **118** 9–12 (in Russian)